Explicit H_∞ controllers for 4^{th} order single-input single-output systems with parameters and their application to the two mass-spring system with damping

Guillaume Rance [*,**,****] **Yacine Bouzidi** [**] **Alban Quadrat** [**] **Arnaud Quadrat** [*] **Fabrice Rouillier** [***]

[*] *Safran Electronics & Defense, 100 Avenue de Paris, Massy, France. (e-mail: {guillaume.rance, arnaud.quadrat}@safrangroup.com).*
[**] *Inria Lille - Nord Europe, Non-A project, 40 Avenue Halley, Bat A - Park Plaza, 59650 Vileneuve d'Ascq, France. (e-mail: {alban.quadrat, yacine.bouzidi}@inria.fr)*
[***] *Inria Paris, Ouragan, Institut de Mathématiques de Jussieu-Paris Rive Gauche, University Pierre and Marie Curie, 4 place Jussieu, 75005 Paris, France (e-mail: fabrice.rouillier@inria.fr).*
[****] *Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CentraleSupélec - CNRS -Université Paris-Sud, 3 rue Joliot-Curie, 91192 Gif-sur-Yvette, France*

Abstract: The purpose of this paper is to explicitly characterize H_∞ controllers for 4^{th} order single-input single-output (SISO) systems in terms of their coefficients considered as unknown parameters. In the SISO case, computing H_∞ controllers requires to find the real positive definite solution of an algebraic Riccati equation (ARE). Due to the system parameters, no purely numerical method can be used to find such a solution, and thus parametric H_∞ controllers. Using elimination techniques for zero-dimensional polynomial systems, we first give a rational parametrization of all the solutions of the ARE. Then, as the problem reduces to solving polynomials of degree 4, closed-form solutions are obtained for all the solutions of this ARE by using expressions by radicals. Using the concept of discriminant variety, we then show that the maximal real root of one of these polynomials is encoded by two different closed-form expressions depending on the values of the system parameters, which yields to different positive definite solution of the ARE. The above results are then used to explicitly compute the H_∞ criterion γ_{opt} and H_∞ controllers in terms of the system parameters. Finally, we study in detail a particular system: the two-mass-spring system with damping. Due to the low number of parameters, we can plot the variations of γ_{opt} in function of the parameters, compute approximations of γ_{opt} at a working point, and derive the expression of a weight function of the parameters to set γ_{opt} to a desired value.

Keywords: Robust control theory, parametric control, linear systems theory, algebraic systems theory, symbolic computation, polynomial methods.

1. INTRODUCTION

In the last decades, robust control theory has played a major role in automatic control by providing methods which take into account uncertainties, model errors, perturbations, etc. in the design of the controllers. One of these methods, called H_∞ control, provides a natural compromise between the performance and the robustness to perturbations and uncertainties of the closed-loop system. In this article, we focus on the H_∞ *loop-shaping robust control problem*, which was firstly introduced in Glover et al. (1989) and then further developed, for instance, in Vinnicombe et al. (2001); Zhou et al. (1996). This problem involves the resolution of an *Algebraic Riccati Equation* (ARE) and eigenvalues calculations, which are both classically done numerically.

Another method consists in studying this problem in a symbolic way, i.e., studying it for a class of systems depicted by some parameters (see Kanno et al. (2012, 2007); Rance et al. (2016-a)). The goal is then to obtain parametric H_∞ controllers for this class of systems. Given a parametric controller for a system with unfixed parameters, only numerical evaluations of these parameters are then required to obtain an H_∞ controller for the system with these fixed values of parameters. This property can be interesting in the design of *adaptive controllers* since such symbolic controllers could easy be embedded. Such a method is also interesting in a design stage of a project to quickly select a good architecture that can satisfy some given specifications. Finally, this paper has also an *informative vocation* since no particular theoretical knowledge is required to use the parametric controllers.

This paper focuses on finding symbolic H_∞ controllers for linear single-input single-output (SISO) systems of order 4 with unknown parameters. The method is based on Rance et al. (2016-a), which provides a symbolic-numeric method for the H_∞ design problem. The resolution of the H_∞ control problem is based on the computation of the positive definite solution X of an ARE (Section 2). Solving this ARE can be reduced to finding the roots of a univariate polynomial $\mathcal{P}$ of order 4 (Section 3), which roots can be found by radicals by means of *Ferrari's formulas* (see, e.g., Tignol (2002)). The study of the *discriminant variety* of $\mathcal{P}$ gives a decomposition of the space of parameters into cells above which the number of real roots of $\mathcal{P}$ is constant (Lazard et al. (2007)). Above each cell, we can identify the solution which is the maximal real root of $\mathcal{P}$, which yields to the positive definite solution X of the ARE (Section 4). Moreover, given X, the computation of the H_∞ criterion γ_{opt} is reduced to the search for the maximal real root of a characteristic polynomial $\mathcal{H}$ of degree 4. Again, we can express the roots of $\mathcal{H}$ by radicals and we can prove that the maximal real root of $\mathcal{H}$ is defined by the same expression by radicals, which directly yields γ_{opt} (Section 5) and thus closed-form formulas of the H_∞ controllers.

Finally, Section 6 illustrates the above approach with a standard Benchmark example, the two-mass-spring-damper system, augmented with a static tuning parameter w. For this system, γ_{opt} admits the same closed-form expression over the entire space of the parameters, except at a singularity. Furthermore, we can approximate γ_{opt} at a working point by a *Taylor expansion* or a *Puiseux expansion*. By using a tuning parameter w, such expansions allows us to ensure a value of γ_{opt} at the given working point. Identifying the physical parameters in real time, we can compute a controller that ensures good stability margins in the neighbourhood of the working point.

2. THE STANDARD H_∞-CONTROL PROBLEM

In this paper, we shall consider 4^{th} order single-input single-output (SISO) finite-dimensional linear systems (Figure 1) defined by their transfer function

$$G := \frac{y_1}{e_1} = \frac{c_3\,s^3 + c_2\,s^2 + c_1\,s + c_0}{s^4 + a_3\,s^3 + a_2\,s^2 + a_1\,s + a_0}\,, \qquad (1)$$

where $a_i,\, c_i \in \mathbb{R}$ for $i = 0, \ldots, 3$. We note $a := (a_0, \ldots, a_3)$ and $c := (c_0, \ldots, c_3)$ the system parameters of (1). We also consider its controllable canonical form defined by:

$$\dot{x} = A\,x + B\,e_1, \quad y_1 = C\,x, \qquad (2)$$

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix} \in \mathbb{R}^{4\times 4},\; B := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{4\times 1},$$

$$C := (\, c_0 \;\; c_1 \;\; c_2 \;\; c_3 \,) \in \mathbb{R}^{1\times 4}.$$

Given a rational controller K, i.e., an element in the field of rational functions with real coefficients $\mathbb{R}(s)$, we consider the closed-loop system defined in Figure 1, and we have

$$\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} S & K\,S \\ G\,S & G\,K\,S \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$S := (1 + G\,K)^{-1}$ being the sensitivity transfer function.

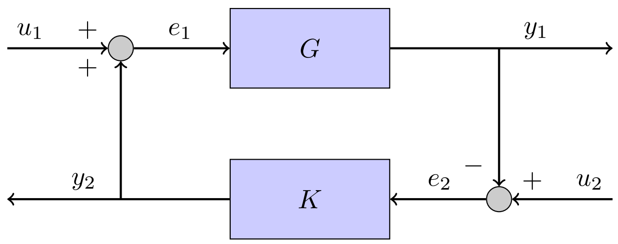

Fig. 1. Control scheme

Let us consider the following standard control problem.

Robust Control Problem (RCP): Given $\gamma > 0$, find a controller K which stabilizes G (i.e., such that the rational transfer functions S, $K\,S$ and $G\,S$ are proper and stable) and is such that:

$$\left\| \begin{pmatrix} S & K\,S \\ G\,S & G\,K\,S \end{pmatrix} \right\|_\infty < \gamma. \qquad (3)$$

A controller K satisfying (3) ensures a compromise between the performance of the closed-loop system and the robustness with respect to the perturbations u_1 and u_2. For more details, see Glover et al. (1989); Zhou et al. (1996); Vinnicombe et al. (2001) and the references therein. The following standard result gives a solution to the RCP.

Theorem 1. (Glover et al., 1989, Cor. 5.1), (Zhou et al., 1996, Ch. 18), Let (A, B, C) be an observable state-space representation (2) of the transfer function G defined by (1). Let X be the unique real positive definite solution of the following ARE

$$\mathscr{R} := X\,A + A^T\,X - X\,B\,B^T\,X + C^T\,C = 0. \qquad (4)$$

Let $Q = Q^T$ be an Hankel matrix defined by

$$Q^{-1} := P = (P_1 \;\cdots\; P_4)\,,$$
$$P_i^{\,T} := C \sum_{j=0}^{4-i} a_{4-j}\, A^{4-i-j}, \quad i = 1, \ldots, 4, \qquad (5)$$

Let $Y := QXQ$. Then, the minimal value of γ, denoted γ_{opt}, such that the RCP admits a solution is given by

$$\gamma_{\mathrm{opt}} := \sqrt{1 + \lambda_{\max}\,(Y X)}.$$

where $\lambda_{\max}$ is the greatest eigenvalue of $Y\,X$ (which one has only real positive eigenvalues). For $\gamma > \gamma_{\mathrm{opt}}$, a controller K_γ satisfying the RCP is defined by

$$\dot{z} = A_\gamma\,z + B_\gamma\,e_2, \quad y_2 = C_\gamma\,z,$$

with the following notations:

$$\begin{cases} Z_\gamma := \left(I + Y\,X - \gamma^2\,I\right)^{-1}, \\ A_\gamma := A - B\,B^T\,X + \gamma^2\,Z_\gamma\,Y\,C^T\,C, \\ B_\gamma := -\gamma^2\,Z_\gamma\,Y\,C^T, \\ C_\gamma := B^T\,X. \end{cases} \qquad (6)$$

Remark 1. Q is related to Kalman's observability matrix $\mathscr{O} := \left(C, CA, CA^2, CA^3\right)^T$ since we have:

$$\Delta_o := \det(\mathscr{O}) = \det(Q^{-1}). \qquad (7)$$

Then, if the system is not observable, Q cannot be computed. For instance, this happens when $a_0 = c_0 = 0$, i.e., s can be factorized in both the numerator and denominator of G. As a consequence, in what follows, we shall suppose that (2) is observable. $\square$

In this paper, for systems of order 4, we focus on the explicit computation of γ_{opt} when the *a_i's and c_j's are unknown parameters and not fixed numerical values.* In particular, *numerical algorithms* cannot be used here. The method follows Algorithm 1 of Rance et al. (2016-a), which combines symbolical and numerical computations to solve the RCP. Since we consider systems of order 4, this algorithm helps us to compute purely symbolic H_∞ controllers satisfying the RCP.

3. PARAMETRIZATION OF ALL THE COMPLEX SOLUTIONS OF THE ARE $\mathscr{R} = 0$

In order to compute a controller K satisfying the above problem, we first have to solve (4). The entries of X, solution of $\mathscr{R} = 0$, are determined only by the b_k's as stated in Theorem 2 of Rance et al. (2016-a):

$$\begin{cases} x_{i,4} = \; b_i - a_i,\; 0 \le i \le 4, \\ x_{1,j} = \; b_0\, b_j - (a_0\, a_j + c_0\, c_j)\,,\; 1 \le j \le 3, \\ x_{2,2} = \; b_1\, b_2 - b_0\, b_3 + a_0\, a_3 + c_0\, c_3 - (a_1\, a_2 + c_1\, c_2), \\ x_{2,3} = \; b_1\, b_3 - b_0 + a_0 - (a_1\, a_3 + c_1\, c_3), \\ x_{3,3} = \; b_2\, b_3 - b_1 - (a_2\, a_3 + c_2\, c_3) + a_1, \end{cases} \quad (8)$$

where the b_k's satisfy the polynomial system $\mathcal{B}$

$$\mathcal{B} := \begin{cases} \mathcal{B}_0 := b_0^2 - d_0 = 0, \\ \mathcal{B}_1 := b_1^2 - 2\, b_0\, b_2 - d_2 = 0, \\ \mathcal{B}_2 := b_2^2 - 2\, b_1\, b_3 + 2\, b_0 - d_4 = 0, \\ \mathcal{B}_3 := b_3^2 - 2\, b_2 - d_6 = 0, \end{cases} \quad (9)$$

and the d_i's are defined by:

$$\begin{cases} d_0 := a_0^2 + c_0^2, \\ d_2 := a_1^2 + c_1^2 - 2\,(a_0\, a_2 + c_0\, c_2)\,, \\ d_4 := a_2^2 + c_2^2 - 2\,(a_1\, a_3 + c_1\, c_3) + 2\, a_0, \\ d_6 := a_3^2 + c_3^2 - 2\, a_2. \end{cases}$$

We want to find closed-form solutions of $\mathcal{B}$. From $\mathcal{B}_3 = 0$, we first obtain

$$b_2 = \frac{1}{2}\left(b_3^2 - d_6\right), \quad (10)$$

which, by substitution into $\mathcal{B}_2 = 0$, then yields:

$$b_1 = \left(b_3^4 - 2\, d_6\, b_3^2 + 8\, b_0 + d_6^2 - 4\, d_4\right) / \left(8\, b_3\right). \quad (11)$$

Substituting (11) and (10) into $\mathcal{B}_1 = 0$, we get:

$$\begin{aligned} \mathcal{P}(b_3) := \; & b_3^8 - 4\, d_6\, b_3^6 + \left(6\, d_6^2 - 8\, d_4 - 48\, b_0\right) b_3^4 \\ & +4\left(-d_6^3 + 4\,(d_4 + 2\, b_0)\, d_6 - 16\, d_2\right) b_3^2 \\ & + \left(d_6^2 - 4\, d_4 + 8\, b_0\right)^2. \end{aligned} \quad (12)$$

Hence, $\mathscr{R} = 0$ admits 16 complex solutions defined by

$$\begin{cases} b_0 = \pm\sqrt{d_0}, \\ b_1 = \left(b_3^4 - 2\, d_6\, b_3^2 + 8\, b_0 + d_6^2 - 4\, d_4\right) / \left(8\, b_3\right), \\ b_2 = \left(b_3^2 - d_6\right)/2, \end{cases} \quad (13)$$

where b_3 satisfies the polynomial equation $\mathcal{P}(b_3) = 0$ of degree 8 (see (12)).

4. POSITIVE DEFINITE SOLUTION OF $\mathscr{R} = 0$

Given $\mathcal{P}$, we want to find the root which yields the positive definite solution of $\mathscr{R} = 0$. Kanno et al. (2009) shows that $X > 0$ is obtained by choosing the maximal real root of $\mathcal{P}$. According to Proposition 5 of Rance et al. (2016-a), note also that $X > 0$ satisfies $b_0 = \sqrt{d_0}$. To express the roots of $\mathcal{P}$, we introduce the following notations:

$$\begin{cases} p := -8\,(6\, b_0 + d_4)\,, \\ q := -64\,(d_2 + b_0\, d_6)\,, \\ r := 16\,(2\, b_0 - d_4)^2 \\ \quad\;\; - 64\, d_2\, d_6, \\ p_2 := -\left(\dfrac{p^2}{12} + r\right), \\ q_2 := \dfrac{p}{3}\left(r - \dfrac{p^2}{36}\right) - \dfrac{q^2}{8}, \end{cases} \begin{cases} \varepsilon_1 := \pm 1,\; \varepsilon_2 := \pm 1 \\ \varepsilon := (\varepsilon_1, \varepsilon_2), \\ \delta_2 := 27\left(4\, p_2^3 + 27\, q_2^2\right), \\ \alpha_2 := \left(\dfrac{\sqrt{\delta_2} - 27\, q_2}{2}\right)^{\frac{1}{3}}, \\ u_2 := \dfrac{1}{3}\left(\alpha_2 - \dfrac{3\, p_2}{\alpha_2} - p\right), \\ \Delta_2 := -\left(u_2 + p + \dfrac{\varepsilon_1\, q}{\sqrt{2\, u_2}}\right). \end{cases} \quad (14)$$

The polynomial $\mathcal{P}$ is even. Finding its roots is equivalent to finding the roots of the following univariate polynomial

$$\widetilde{\mathcal{P}}(t) = t^4 + p\, t^2 + q\, t + r, \quad (15)$$

where $t := b_3^2 - d_6$. Since $\widetilde{\mathcal{P}}$ is a polynomial of degree 4, its roots can be expressed by means of radicals (Ferrari's formula) as follows (see, e.g., Tignol (2002)):

$$\begin{cases} \text{if } q \ne 0 \,:\, t(\varepsilon) := \dfrac{\sqrt{2}}{2}\left(\varepsilon_1\sqrt{u_2} + \varepsilon_2\sqrt{\Delta_2}\right), \\ \text{if } q = 0 \,:\, t(\varepsilon) := \dfrac{\sqrt{2}}{2}\varepsilon_1\sqrt{\varepsilon_2\sqrt{p^2 - 4\, r} - p}. \end{cases} \quad (16)$$

The following notations will be used below to find the maximal real root of $\mathcal{P}$:

$$\Delta_{\widetilde{\mathcal{P}}} := r^3 + \beta_2\, r^2 + \beta_1\, r + \beta_0, \quad (17)$$

$$\beta_2 := \frac{-p^2}{2},\; \beta_1 := \frac{p}{16}\left(p^3 + 9\, q^2\right),\; \beta_0 := \frac{-q^2}{64}\left(\frac{27}{4}q^2 + p^3\right),$$

$$\begin{cases} p_3 := -\dfrac{1}{3}\beta_2^2 + \beta_1, \\ q_3 := \dfrac{2\,\beta_2^3}{27} - \dfrac{\beta_1\beta_2}{3} + \beta_0 \\ \delta_3 := 27\left(4\, p_3^3 + 27\, q_3^2\right), \\ \alpha_3 := \left(\dfrac{\sqrt{\delta_3} - 27\, q_3}{2}\right)^{\frac{1}{3}}, \end{cases} \begin{cases} j := \left(-1 + i\sqrt{3}\right)/2, \\ r_1 := \dfrac{1}{3}\left(\dfrac{\alpha_3^2 - 3\, p_3}{\alpha_3} - \beta_2\right), \\ r_2 := \dfrac{1}{3}\left(\dfrac{\alpha_3^2 - 3\, p_3}{\alpha_3\, j} - \beta_2\right), \\ r_3 := \dfrac{1}{3}\left(\dfrac{\alpha_3^2 - 3\, p_3}{\alpha_3\, j^2} - \beta_2\right). \end{cases} \quad (18)$$

Note that the r_i's are the roots of $\Delta_{\widetilde{\mathcal{P}}}(r)$, the discriminant of $\widetilde{\mathcal{P}}$ (see Tignol (2002)). We also note $q_0 := \sqrt{-8\, p^3/27}$. The following proposition gives the maximal real root of $\widetilde{\mathcal{P}}$ depending on the value of the parameters.

Proposition 1. Suppose that $\widetilde{\mathcal{P}}$, defined in (15), has at least one real root, i.e. (2) is observable. Assuming $q \ne 0$, $t(1,1)$ (see (16)) is the greatest real root of $\widetilde{\mathcal{P}}$ if and only if (p, q, r) (see (14)) satisfies one of the following conditions

$$\begin{cases} C_1 := \{p \le 0,\; q \in [-q_0\,;\, 0[\,,\; r \in]-\infty\,;\, r_3] \cup [r_2\,;\, r_1]\}\,, \\ C_2 := \{p \le 0,\; q \le q_0,\; r \le r_1\}\,, \\ C_3 := \{p \le 0,\; q \in [-q_0\,;\, q_0] \setminus \{0\}\,,\; r \in [r_3\,;\, r_2]\}\,, \\ C_4 := \{p \ge 0,\; q < 0,\; r \le r_1\}\,, \end{cases} \quad (19)$$

and $t(1,-1)$ (see (16)) is the greatest real root of $\widetilde{\mathcal{P}}$ if and only if (p, q, r) satisfies one of the following conditions:

$$\begin{cases} C_5 := \{p \le 0,\; q \in]0\,;\, q_0]\,,\; r \in]-\infty\,;\, r_3] \cup [r_2\,;\, r_1]\}\,, \\ C_6 := \{p \le 0,\; q \ge q_0,\; r \le r_1\}\,, \\ C_7 := \{p \ge 0,\; q > 0,\; r \le r_1\}\,. \end{cases} \quad (20)$$

Assuming $q = 0$, $t(1,1)$ is the maximal real root of $\widetilde{\mathcal{P}}$.

Proof. The proof is based on the concept of *discriminant variety* (Lazard et al. (2007)). Given an open connected set in the space of parameters which does not encounter the discriminant variety of $\mathcal{P}$, for any values of the parameters in this set, $\mathcal{P}$ has a constant number of real roots. Over the discriminant variety, some roots are crossing, i.e., 2 closed-form solutions can define the same maximal real root. In this case, the discriminant variety of $\mathcal{P}$ equals its discriminant $\Delta_{\widetilde{\mathcal{P}}}$ as $\mathcal{P}$ is monic. The reader is referred to Rance et al. (2016-b) for a detailed proof.

Let $\mathcal{E} := \{(-1,-1),(-1,1),(1,-1),(1,1)\}$. We note $t_{\max} := \max_{\varepsilon\in\mathcal{E}} \{t(\varepsilon) \mid t(\varepsilon)\in\mathbb{R}\}$ the greatest real root of $\widetilde{\mathcal{P}}$. Then, we denote by σ the maximal real root of $\mathcal{P}$, i.e., $\sigma := \sqrt{t_{\max}+d_6}$. Thus, we have:

$$\sigma = \begin{cases} \sqrt{t(1,1)+d_6}, & (p,q,r)\in C_i,\ 1\le i\le 4,\\ \sqrt{t(1,-1)+d_6}, & (p,q,r)\in C_i,\ 5\le i\le 7,\\ \sqrt{t(1,1)+d_6}, & q=0. \end{cases}$$

where t is defined in (16). The solution of (9) corresponding to $X>0$ is of the form:

$$\begin{cases} b_0 = \sqrt{d_0},\\ b_1 = \dfrac{\sigma^4 - 2\,d_6\,\sigma^2 + 8\,b_0 + d_6^2 - 4\,d_4}{8\,\sigma}, \end{cases} \quad \begin{cases} b_2 = \dfrac{\sigma^2 - d_6}{2},\\ b_3 = \sigma. \end{cases}$$

Then, (8) is used to obtain explicitly $X>0$.

5. COMPUTING γ_{OPT} AND H_∞ CONTROLLERS

The Hankel matrix Q (see (5)) is of the form $Q = \Delta_o^{-1}\,Q_r$ (see (7)), where Q_r is also an Hankel matrix. Given Q, we can compute $Y := QXQ$. Then, the characteristic polynomial of YX is of the form:

$\mathcal{H}(\lambda,a,c) = \lambda^4 + \nu_3(a,c)\,\lambda^3 + \nu_2(a,c)\,\lambda^2 + \nu_1(a,c)\,\lambda + \nu_0(a,c)$.

where ν_i's are coefficients depending on a and c. Note that Q_r, X, Y, Δ_o and ν_i's can all be computed in terms of a and c using a symbolic computing environment such as `Maple`. Let us now introduce the following notations:

$$\begin{cases} \varepsilon_1 := \pm 1, \varepsilon_2 := \pm 1,\\ \varepsilon := (\varepsilon_1,\varepsilon_2),\\ \mu_2 := -\dfrac{3}{8}\,\nu_3^2 + \nu_2,\\ \mu_1 := \dfrac{\nu_3^3}{8} - \dfrac{\nu_2\,\nu_3}{2} + \nu_1,\\ \mu_0 := \dfrac{\nu_3^2}{16}\left(\nu_2 - \dfrac{3\,\nu_3^2}{16}\right)\\ \quad -\dfrac{1}{4}\,\nu_1\,\nu_3 + \nu_0, \end{cases} \quad \begin{cases} p_4 := -\left(\mu_2^2/12 + \mu_0\right),\\ q_4 := \dfrac{\mu_2}{3}\left(\mu_0 - \dfrac{\mu_2^2}{36}\right) - \dfrac{\mu_1^2}{8},\\ \delta_4 := 27\left(4\,p_4^3 + 27\,q_4^2\right),\\ \alpha_4 := \left(\dfrac{\sqrt{\delta_4} - 27\,q_4}{2}\right)^{\frac{1}{3}},\\ u_4 := \dfrac{1}{3}\left(\alpha_4 - \dfrac{3\,p_4}{\alpha_4} - \mu_2\right),\\ \Delta_4 := -\left(u_4 + \mu_2 + \dfrac{\varepsilon_1\,\mu_1}{\sqrt{2\,u_4}}\right). \end{cases} \quad (21)$$

Since $\mathcal{H}$ is a polynomial of degree 4, we can obtain its roots by radicals as explained, e.g., in Tignol (2002):

$$\begin{cases} \text{if } \mu_1 \neq 0\colon\ \lambda(\varepsilon) := \dfrac{\sqrt{2}}{2}\left(\varepsilon_1\sqrt{u_4} + \varepsilon_2\sqrt{\Delta_4}\right) - \dfrac{\nu_3}{4},\\ \text{if } \mu_1 = 0\colon\ \lambda(\varepsilon) := \dfrac{\sqrt{2}}{2}\varepsilon_1\sqrt{\varepsilon_2\sqrt{\mu_2^2 - 4\,\mu_0} - \mu_2}. \end{cases} \quad (22)$$

Since YX is the product of two positive definite matrices, the roots of $\mathcal{H}$ are all real strictly positive. The following proposition, which is a consequence of Proposition 1 in the case where the polynomial under study as only real roots, helps us to find which root is the greatest one.

Proposition 2. Given $\lambda(\varepsilon)$ as defined in (22), the expression by radicals of the maximal real root of $\mathcal{H}$ is

$$\lambda_{\max}(YX) = \max_{\varepsilon\in\mathcal{E}} \{\lambda(\varepsilon) \mid \lambda(\varepsilon)\in\mathbb{R}\} = \lambda(1,1).$$

Then, Proposition 2 yields $\lambda_{\max} = \lambda(1,1)$. From the expression of $\lambda_{\max}$, we deduce γ_{opt} as follows:

$$\gamma_{\mathrm{opt}} = \sqrt{1+\lambda(1,1)}. \quad (23)$$

Using the results of Section 4 giving $X>0$ depending on the parameters, and using (6) of Theorem 1, we can deduce (sub)-optimal H_∞ controllers K_γ.

6. A STANDARD EXAMPLE: THE TWO-MASS-SPRING-DAMPER SYSTEM

6.1 Problem under consideration

We illustrate the above approach with the model of a *two-mass-spring-damper system* (see Figure 2) considered in Wie et al. (1992); Vinnicombe et al. (2001); Alazard et al. (1999). The latter system is a standard benchmark in robust control theory.

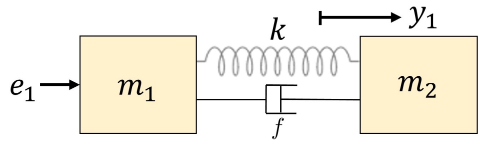

Fig. 2. The two-mass-spring-damper system

Two masses m_1 and m_2 are linked by a spring of stiffness k and a damper of magnitude f. With the notations of Figure 2, we study the displacement y_1 of m_2, while m_1 is excited by a force e_1. We consider the transfer function of the *physical plant* from the input e_1 to the output y_1:

$$\frac{y_1}{e_1} := P = \frac{a_3\,s + a_2}{m\,s^2\,(s^2 + a_3\,s + a_2)},\ m := m_1 + m_2 > 0,$$

$$a_2 := \frac{(m_1+m_2)\,k}{m_1\,m_2} > 0,\ \ a_3 := \frac{(m_1+m_2)\,f}{m_1\,m_2} > 0.$$

As in (Vinnicombe et al., 2001, §2.6, §4), we consider a static weight w as $e_1 = w\,\tilde{e}_1$ and define the *fictive plant* by:

$$\frac{y_1}{\tilde{e}_1} := G = w\,P = \frac{w}{m\,s^2}\,\frac{a_3\,s + a_2}{(s^2 + a_3\,s + a_2)}. \quad (24)$$

Given a robust controller K_γ stabilizing G, we get a robust controller $C_\gamma := w\,K_\gamma$ stabilizing P since $K_\gamma\,G = C_\gamma\,P$. This controller C_γ only satisfies $\|S\|_\infty < \gamma$ and $\|G\,K_\gamma\,S\|_\infty < \gamma$ (Vinnicombe et al., 2001, Cor. 5.1). Note that the weight w modifies the norms $\|K_\gamma\,S\|_\infty$ and $\|G\,S\|_\infty$, but provides a degree of freedom that will be used later on to fix γ_{opt} to a desired value. Hence, using the results previously developed, we focus on computing an explicit controller stabilizing G.

In this example, the problem involves only 4 parameters

$$\theta := (a_2,\,a_3,\,m,\,w),$$

and the d_{2i}'s are

$$d_0 = \frac{w^2\,a_2^2}{m^2},\ d_2 = \frac{w^2\,a_3^2}{m^2},\ d_4 = a_2^2,\ d_6 = a_3^2 - 2\,a_2, \quad (25)$$

which highly simplifies the computation of the solution $X>0$ of $\mathscr{R}=0$ and γ_{opt} as shown in the next paragraph.

6.2 Solution of the RCP

This example implies a fewer number of parameters than in the general previous context. As a consequence, and using again the concept of discriminant variety (as in Proposition 1), we show that the maximal real root of $\mathcal{P}$ has the same closed-form expression over the entire space of the parameters θ (except at $q = 0$):

$$\sigma = \sqrt{t(1,1) + d_6},$$

where $t(1,1)$ is defined in (16), and yields $X > 0$. Using (5), we compute the matrix Q, and thus $Y := Q\,X\,Q$. We then compute the characteristic polynomial $\mathcal{H}$ of YX, and write its roots as in (22). Then, using (23), we deduce $\gamma_{\mathrm{opt}} = \sqrt{1 + \lambda(1,1)}$, We note that γ_{opt} only depends on the two variables G_r and ρ defined as follows:

$$G_r := \frac{w}{m\,a_2}, \qquad \rho := \frac{a_3}{\sqrt{a_2}}. \tag{26}$$

A plot of γ_{opt} depending on G_r and ρ is given in Figure 3.

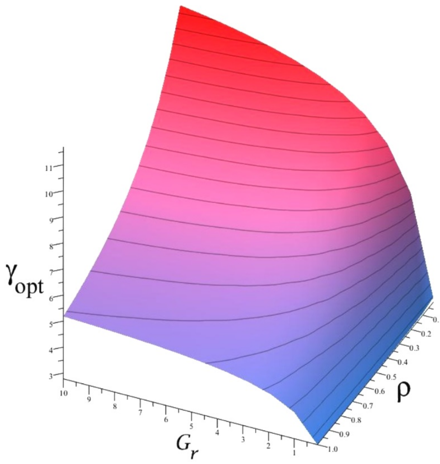

Fig. 3. Plot of γ_{opt} in function of G_r and ρ.

Using Theorem 1, we can deduce a (sub-)optimal H_∞ controller K_γ stabilizing G, which yields a (sub-)optimal stabilizing controller $C_\gamma := w\,K_\gamma$ of the physical plant P.

6.3 Setting γ_{opt} to a desired value

In the previous section, we have found an explicit formula (23) of γ_{opt} depending on G_r and ρ. In practice, engineers want to ensure some performance and robustness properties for a given configuration. Vinnicombe et al. (2001) provides a link between $\gamma \geq \gamma_{\mathrm{opt}}(G_r, \rho)$ and *guaranteed gain and phase margins* δ_G and δ_Φ:

$$\begin{cases} \Delta_{\mathrm{G}}(G, K_\gamma) \geq \delta_G(\gamma) := \dfrac{1 + \gamma^{-1}}{1 - \gamma^{-1}}, \\ \Delta_\Phi(G, K_\gamma) \geq \delta_\Phi(\gamma) := 2 \arcsin\left(\gamma^{-1}\right), \end{cases} \tag{27}$$

where Δ_G (resp. Δ_Φ) represents the *gain* (resp. *phase*) *margin* of the open-loop. In practice, a good value for $\gamma_{\mathrm{opt}}(G_r, \rho)$ is 3, which ensures good stability margins (i.e., $\Delta_G \geq 6\,\mathrm{dB}$, $\Delta_\Phi \geq 39°$). Then, in this section, we study how to set w to ensure $\gamma_{\mathrm{opt}}(G_r, \rho) = \bar{\gamma}_{\mathrm{opt}}$, where:

$$\bar{\gamma}_{\mathrm{opt}} := 3.$$

In this perspective, we study the algebraic variety defined by the polynomial expression of γ_{opt} in terms of the parameters G_r and ρ.

Study of the algebraic varieties L and $\bar{L}$. In order to compute γ_{opt}, we have to choose $\lambda_{\max}$ the maximal real λ satisfying $\mathcal{H}(\lambda, \sigma) = 0$, while σ is the maximal real root of $\mathcal{P}$ in b_3. The following system

$$\begin{cases} \mathcal{P}(b_3, \theta) = 0, \\ \mathcal{H}(\lambda, b_3, \theta) = 0, \end{cases} \tag{28}$$

parametrizes all the eigenvalues of all the possibles matrices $YX = QXQX$, where X is a solution of $\mathscr{R} = 0$. As only λ is of interest here (because it directly leads to $\gamma_{\mathrm{opt}} = \sqrt{1 + \lambda(1,1)}$), instead of (28), we can consider the projection of (28) onto the variable λ:

$$\mathcal{L}(\lambda, \theta) = 0.$$

To compute the polynomial $\mathcal{L}$, we have to eliminate b_3 from (28) by computing, for instance, the *resultant* of $\mathcal{H}$ and $\mathcal{P}$ for the variable b_3 (see, e.g., Chapter 3 of Cox et al. (2005)). The polynomial $\mathcal{L}$ contains $\lambda = \lambda_{\max}$ as well as the other λ which correspond to non positive definite matrices (i.e., non real maximal b_3), or to non maximal real λ.

Since $\mathcal{P}$ is of degree 8 in b_3 and $\mathcal{H}$ is of degree 4 in λ, $\mathcal{L}$ is of degree 32 in λ. Thus, $\mathcal{L}$ is too long to be printed here. We can also find again that $\mathcal{L}(\lambda, \theta)$ only depends on the two parameters G_r and ρ defined in (26). With these changes of variable, we obtain:

$$L(\lambda, G_r, \rho) = 0$$

A plot of L for $\rho \in \{0,\ 1/100,\ 1/10\}$ is given in Figure 4.

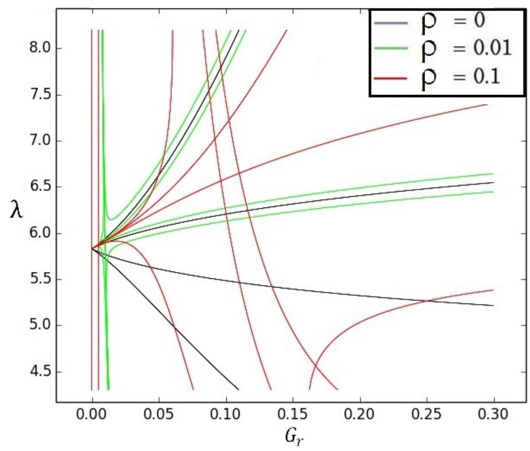

Fig. 4. $L(\lambda, G_r, \rho) = 0$ for $\rho \in \{0,\ 1/100,\ 1/10\}$

Rouillier et al. (2016) shows an animated drawing of L where ρ varies from 0 to 3/4. Since we are only interested in values of (G_r, ρ) yielding to $\gamma_{\mathrm{opt}} = \bar{\gamma}_{\mathrm{opt}}$, i.e., to $\lambda_{\max} = \bar{\lambda}_{\max}$ where

$$\bar{\lambda}_{\max} := \bar{\gamma}_{\mathrm{opt}}^2 - 1 = 8,$$

we define $\bar{L}(G_r, \rho) := L(\bar{\lambda}_{\max}, G_r, \rho)$. Figure 5 represents $\bar{L} = 0$, which consists in a series of branches. On each branch, an eigenvalue of all the possible matrices YX is equal to $\bar{\lambda}_{\max}$. Thus, we have to identify the one corresponding to $\lambda(1,1)(G_r, \rho) = \bar{\lambda}_{\max}$ in all of them. This task can be done by computing the discriminant variety (see proof of Proposition 1) of $\bar{L}$, which encodes all the critical points of $\bar{L}$ and splits the space $\{G_r, \rho\}$ into cells above which the roots of $\bar{L}$ in ρ are not crossing. Picking a point in each cell, we identify the curve corresponding to $\lambda(1,1)(G_r, \rho) = \bar{\lambda}_{\max}$ as the green one in Figure 5.

Approaching $\bar{L}$ $\bar{L}(G_r, \rho) = 0$ is not solvable explicitly in ρ. However, approaching this curve around a working point (using a Taylor or Puiseux expansion for example), we can get an explicit formula of G_r depending on ρ, which associated curve coincides with $\bar{L}(G_r, \rho)$ in the neighbourhood of the working point. For example, a Puiseux expansion of order 3 around $\bar{\rho} := 1$, gives (red curve in Figure 5):

$$\widehat{G}_r(\rho) := 0.033\ (\rho - 1.0)^2 + 0.11\,\rho + 0.13$$

Then, we set the tuning parameter w as follows:

$$w = \widehat{w} := m\, a_2\, \widehat{G}_r(\rho).$$

Using this w, we ensure that γ_{opt} is close to $\bar{\gamma}_{\text{opt}}$ around our working point. Noting $\gamma := \sqrt{\lambda + 1}$, we can verify the value of γ_{opt} by plotting $L(\gamma^2 - 1, \widehat{G}_r(\rho), \rho) = 0$ (Figure 6). Then, given ρ around $\bar{\rho}$, we can compute $\gamma_{\text{opt}}(\widehat{G}_r(\rho), \rho)$ using (23), choose $\gamma > \gamma_{\text{opt}}(\widehat{G}_r(\rho), \rho)$ and obtain an auto-tuned controller $C_\gamma(\theta) = K_\gamma(\theta)/\widehat{w}$ stabilizing P and ensuring $\gamma_{\text{opt}}(\widehat{G}_r(\rho), \rho)$ close to $\bar{\gamma}_{\text{opt}}$.

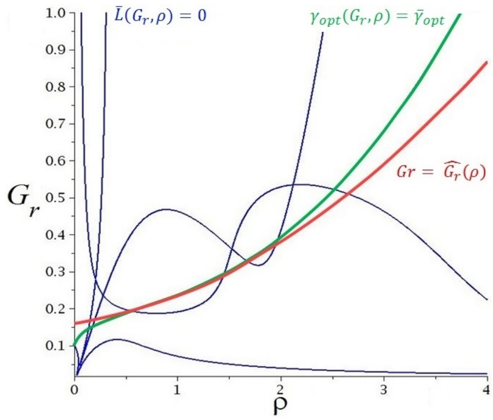

Fig. 5. Blue: $\bar{L}(G_r, \rho) = 0$ – Green: branch corresponding to $\gamma_{\text{opt}}(G_r, \rho)$ – Red: $G_r = \widehat{G}_r(\rho)$.

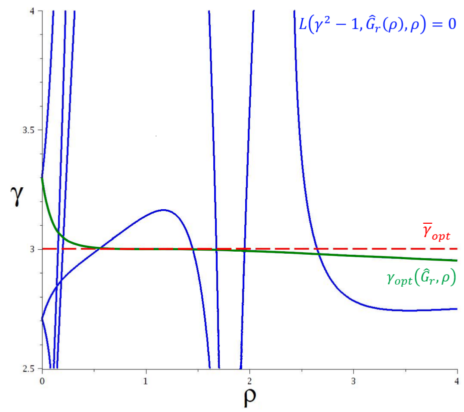

Fig. 6. Blue: $L(\gamma^2 - 1, \widehat{G}_r(\rho), \rho) = 0$ – Green: branch corresponding to $\gamma_{\text{opt}}(\widehat{G}_r(\rho), \rho)$ – Red: Expected γ_{opt}.

7. CONCLUSION

In this article, we gave explicit H_∞ controllers for SISO systems of order 4. We also gave a closed-form expression of γ_{opt}, the H_∞ criterion satisfied by the closed-loop. By introducing a tuning parameter w and approximating γ_{opt} around a working point, we showed how to set γ_{opt} to a desired value. With on-line identification of the physical parameters, we obtain an adaptive controller that ensures good stability margins for any values of the parameters in the neighbourhood of the working point.

To avoid the use of the complicated closed-form expression of γ_{opt} in real time, we must ensure that the real γ_{opt} must not exceed the expected $\bar{\gamma}_{\text{opt}}$ (to avoid creating destabilizing controllers). Therefore, future work will focus on the study of the variations of γ_{opt} depending on the parameters, or other ways to ensure $\gamma_{\text{opt}} > \bar{\gamma}_{\text{opt}}$. Future work will also consider the estimation of the degradation of stability margins and problems brought by uncertain estimations of the parameters.

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